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New Criteria for Meromorphic Bazilević Functions Associated with Linear Operator

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Abstract

In this work, was proposed and characterized a new linear operator and having this linear operator establishing, we benefited from it to define class of meromorphic Bazilević functions in the punctured unit disk $\mathcal{U}^* = \mathcal{U}\backslash\{0\} = \{z \in \mathbb{C}: 0 < |z| < 1\}$. What's more, we obtain some adequate conditions for functions having a place with this class.

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Keywords: *Meromorphic; Bazilević functions; Linear Operator.*

1. **INTRODUCTION**

Let M^*refer the class of all meromorphic functions as the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (n \in \mathbb{N} = \{1,2,\dots\}) \qquad (1)$$

which are analytic and normalized in the punctured unit disk $\mathcal{U}^* = \mathcal{U}\backslash\{0\} = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

The generalized hypergeometric and meromorphic functions were thought about as of late by (Liu and Srivastava[1], Cho and Kim [3] and Dziok and Srivastava [5][6]).

Now, we define the following linear derivative operator

$\Omega_\lambda^{m,\beta}(\alpha,\xi): M^* \to M^*$ as follows:

$$\Omega_\lambda^{0,\beta}(\alpha,\xi)f(z) = f(z),$$

$$\Omega_\lambda^{1,\beta}(\alpha,\xi)f(z) =$$

$$\left(1 - \frac{\beta(\lambda-\alpha)}{\xi+\lambda}\right)\Omega_\lambda^{0,\beta}(\alpha,\xi)f(z) - \left(\frac{\beta(\lambda-\alpha)}{\xi+\lambda}\right) z\left(\Omega_\lambda^{0,\beta}(\alpha,\xi)f(z)\right)'$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left(1 + \frac{\beta(\lambda-\alpha)(n-1)}{\xi+\lambda}\right) a_n z^n,$$

$$\Omega_\lambda^{2,\beta}(\alpha,\xi)f(z) = \left(1 - \frac{\beta(\lambda-\alpha)}{\xi+\lambda}\right)(\Omega_\lambda^{1,\beta}(\alpha,\xi)f(z)) - \left(\frac{\beta(\lambda-\alpha)}{\xi+\lambda}\right) z(\Omega_\lambda^{1,\beta}(\alpha,\xi)f(z))'$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left(1 + \frac{\beta(\lambda-\alpha)(n-1)}{\xi+\lambda}\right)^2 a_n z^n.$$

For general

$$\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z) = \Omega_\lambda^{1,\beta}(\alpha,\xi)\left(\Omega_\lambda^{m-1,\beta}(\alpha,\xi)f(z)\right), \qquad (2)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left(1 + \frac{\beta(\lambda-\alpha)(n-1)}{\xi+\lambda}\right)^m,$$

where $m \in \mathbb{N}_0 = \{0,1,2,\dots\}, \beta \ge 0, \alpha \ge 0, \lambda > 0, \xi > 0,$ and $z \in \mathcal{U}^*$.

It should be noted that, the linear operator $\Omega_\lambda^{m,\beta}(\alpha,\xi)$ generalised many operators studied by several earlier authors by specializing the parameters in this operator.

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If $\xi = 1, \lambda = \beta = 1$ and $\alpha = 0$, the operator $\Omega_1^{m,1}(0,0)$ reduces to operator presented by Sălăgean [13].

If $\xi = 1, \lambda = 1$ and $\alpha = 0$, the operator $\Omega_1^{m,\beta}(0,0)$ reduces to generalized Salagean derivative operator which was presented by Al-Oboudi [10].

Now, we present another subclass $\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\gamma)$ of analytic functions including the linear multiplier operator $\Omega_\lambda^{m,\beta}(\alpha,\xi)$ defined by [1].

Definition 1.1: A function $f(z) \in M^*$ be meromorphic Bazilević of order γ and type μ if it satisfies the following inequality

$$-Re\left\{\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^\mu}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)}\right\} > \gamma, \qquad (3)$$

for $m \in \mathbb{N}_0$, $\beta \geq 0, \alpha \geq 0, \lambda \geq 0$, $\xi \geq 0, 0 \leq \gamma < 1, 0 \leq \mu < 1$ and for all $z \in \mathcal{U}^*$. We denote the former class of functions as $\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\gamma)$.

Based on the Definition 1.1, we get the following remark:

Remark 1.2 It ought to be commented that the class $\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\gamma)$ is a generalization of more classes thought about before. By giving explicit qualities to m and μ for this class, we acquire the accompanying subclasses :

i. If $\mu = 0$ and $m = 0$ in the class $\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\gamma)$, then we have

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma,$$

it reduces to the class $M_{S^*(\gamma)}$ introduced by [4].

ii. If $m = 0$ in the class $\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\gamma)$, then we have

$$-Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\mu\right\} > \gamma,$$

it reduces to the class $B(\mu,\gamma)$ introduced by[2].

Lemma 1.3 [9] Let w be analytic function in unit disk $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains it is maximum quantity on $|z| = r < 1$ at $z_0 \in \mathcal{U}$, then $z_0 w'(z_0) = k w(z_0)$, where $k \in \mathbb{R}$ and $k \geq 1$.

Lemma 1.4. [12] Let $S \subset \mathbb{C}$ and assume that $\varphi(z): \mathbb{C}^2 \times \mathcal{U} \longrightarrow \mathbb{C}$ satisfies $\Phi(ix,y;z) \notin S$ for all $z \in \mathcal{U}$, and for all $x, y \in \mathbb{R}$ such that $y \leq -(1+x^2)/2$. If $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ is analytic function and $\varphi(q(z), zq'(z); z) \in S$ for all $z \in \mathcal{U}$, then $Re\{q(z)\} > 0$.

Lemma 1.5 [8] Let h be analytic function with $h(0) = 1$. Suppose that there exists $z_0 \in \mathcal{U}$ such that $Re\{h(z)\} > 0$ $(|z| < |z_0|)$, $Re\{h(z_0)\} = 0$ and $h(z) \neq 0$. Then we have $h(z) = ia(a \neq 0)$ and

$$\frac{z_0 h'(z_0)}{h(z_0)} = i\frac{n}{2}\left(a + \frac{1}{a}\right),$$

where n is a real number with $n \geq 1$.

2. RESULTS

Theorem 2.1 Let $0 \leq \delta < 1$ and $0 \leq \mu < 1$. If $f(z) \in M^*$ satisfies the following inequality

$$Re\{\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^\mu}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)}[\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^\mu}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)}$$

$$+\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)} - \frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)''}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'} - \mu\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}]\} > \delta(\delta + \frac{1}{2}) + (\delta(\mu-1) - \frac{1}{2}).$$

Then $f(z) \in \mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\delta)$.

Proof. We define the function $q(z)$ by

$$\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^\mu}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)} = -\delta + (\delta - 1)q(z), \qquad (4)$$

Such that $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ is analytic function. Now differentiating logarithmically of (4) with respect to z, we obtain

$$(\delta + (1-\delta)q(z))(\frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))''}{(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'} + \mu\frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)} - \frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)})$$

$$= (\mu - 1)\big(\delta + (1-\delta)q(z)\big) + (1-\delta)zq'(z). \qquad (5)$$

From (4) and (5), we get

$$\left(-\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^\mu}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)}\right)\left(\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)''}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'} + \mu\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)} - \frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}\right)$$

$$= (\mu - 1)(\delta + (1-\delta)q(z)) + (1-\delta)zq'(z). \qquad (6)$$

By using the same technique in (6), we get

$$\left(\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^\mu}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)}\right)^2\left(-\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^\mu}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)}\right)$$

$$\left(\frac{z\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)''}{\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'}+\mu\frac{z\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}-\frac{z\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}\right)$$

$$=(1-\mu)zq'(z)+(1-\delta)^2q^2(z)+(1-\delta)[2\delta+(\mu-1)]q(z)+\delta^2+\delta(\mu-1) \quad (7)$$

$$=\varphi(q(z),zq'(z);z),$$

where

$$\varphi(r,s;z)=(1-\mu)s+(1-\delta)^2r^2+(1-\delta)[2\delta+(\mu-1)]r+\delta^2+\delta(\mu-1).$$

For all real numbers x and y satisfying $y\le -(1+x^2)/2$, we have

$$Re(\varphi(ix,y;z))=(1-\mu)y-(1-\delta)^2x^2+\delta^2+\delta(\mu-1)$$

$$\le -\frac{1}{2}(1-\mu)(1+x^2)-(1-\delta)^2x^2+\delta^2+\delta(\mu-1)$$

$$=-\tfrac{1}{2}(1-\delta)-(1-\delta)(\tfrac{1}{2}+(1-\delta))x^2+\delta(\mu-1)+\delta^2$$

$$\le \delta(\mu-1)+\delta^2-\tfrac{1}{2}(1-\delta)$$

$$=\delta(\delta+\frac{1}{2})+(\delta(\mu-1)-\frac{1}{2}).$$

Let $S=\{w: Re(w)>\delta(\delta+\frac{1}{2})+(\delta(\mu-1)-\frac{1}{2})\}$.

Then $\varphi(q(z),zq'(z);z)\in S$ and $\varphi(ix,y;z)\notin S$ for all real x and $y<-(1+x^2)/2, z\in\mathcal{U}$. By applying Lemma 1.4, we have $Re(q(z))>0$, that is $f(z)\in\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\gamma)$. The proof is complete.

Putting $\mu=0, m=0$ and $\delta=0$ in Theorem 2.1, we have the following result:

Corollary 2.2: If $f(z)\in M^*$ satisfies the following inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\left[\frac{2zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right]\right\}>-\frac{1}{2},$$

then $f(z)\in\mathcal{R}_{\lambda,\alpha}^{0,\beta}(\xi,0,0)$.

For $\mu=1,\delta=1$ and $m=0$ in Theorem 2.1, gives.

Corollary 2.3: If $f(z)\in M^*$ satisfies the following inequality

$$Re\{(f'(z))^2-zf''(z)\}>1,$$

then $f(z)\in\mathcal{R}_{\lambda,\alpha}^{0,\beta}(\xi,1,1)$.

Theorem 2.4: If $f(z)\in M^*$ and satisfies

$$Re\left\{(1-\mu)\frac{z(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z))'}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}-\frac{z(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z))''}{(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z))'}\right\}<2(1-\mu)-\delta,\quad (z\in\mathcal{U})$$

then

$$-Re\left\{\frac{z^{1-\mu}\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}\right\}>\gamma$$

$$=\frac{1}{1+2(1-\mu)-2\delta},\quad (z\in\mathcal{U})$$

where $0\le\mu<1$ and $(2(1-\mu)-1)/2\le\delta<1-\mu$.

Proof. We define the function $h(z)$ in $\mathcal{U}$ as follows

$$-\frac{z^{1-\mu}\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}=\gamma+(1-\gamma)h(z), \quad (8)$$

with $\gamma=\frac{1}{1+2(1-\mu)-2\delta}$. Then clearly $h(z)$ is analytic in $\mathcal{U}$ with $h(0)=1$ and

$$(1-\mu)\frac{z\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}-\frac{z\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)''}{\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'}$$

$$=(1-\mu)-\frac{(1-\gamma)zh'(z)}{\gamma+(1-\gamma)h(z)}. \quad (9)$$

Supoose there exists $z_0\in\mathcal{U}$ such that $Re\{h(z)\}>0$, $(|z|<|z_0|)$, $Re\{h(z_0)\}=0, h(z)\neq 0$.

Therefore, by applying Lemma 1.5, we have $h(z)=ia\quad(a\neq 0)$, and

$$\frac{z_0h'(z_0)}{h(z_0)}=i\frac{n}{2}\left(a+\frac{1}{a}\right).\quad (n\ge 1)$$

We conclude from this that

$$(1-\mu)\frac{z_0\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)\right)'}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)}-\frac{z\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)\right)''}{\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)\right)'}$$

$$=1-\mu-\frac{(1-\gamma)zh'(z_0)}{\gamma+(1-\gamma)h(z_0)}=1-\mu+\frac{n(1-\gamma)(1+a^2)}{2(\gamma+i(1-\gamma)a)}. \quad (10)$$

Furthermore, we get

$$Re\left\{(1-\mu)\frac{z_0\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)\right)'}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)}-\frac{z\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)\right)''}{\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z_0)\right)'}\right\}$$

$$=1-\mu+\frac{n(1-\gamma)(1+a^2)}{2(\gamma^2+(1-\gamma)^2a^2)} \quad (11)$$

$$\geq 1-\mu+\frac{n(1-\gamma)}{2\gamma}$$

$$\geq 2(1-\mu)-\delta.$$

This goes against our assumption. Hence, $Re\{h(z)\}>0$ for all $z\in\mathcal{U}$. Thus

$$-Re\left\{\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}\right\}>\gamma$$

$$=\frac{1}{1+2(1-\mu)-2\delta},(z\in\mathcal{U}).$$

If putting $\delta=(2(1-\mu)-1)/2$ in Theorem 2.4, we obtain:

Corollary 2.5: If $f(z)\in M^*$ and satisfies the following inequality

$$Re\left\{(1-\mu)\frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}-\frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))''}{(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'}\right\}<\frac{3}{2}-\mu,\quad (z\in\mathcal{U})$$

then

$$-Re\left\{\frac{z^{1-\mu}(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))^{\mu}}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}\right\}>\gamma=\frac{1}{2},\qquad (z\in\mathcal{U})$$

where $0\leq\mu<1$.

For $\mu=0$ and $m=0$ in Theorem 2.4, we get:

Corollary 2.6: If $f(z)\in M^*$ and satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}-\frac{zf''(z))}{f'(z)}\right\}<2-\delta,\quad (z\in\mathcal{U})$$

then

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\}>\gamma=\frac{1}{3-2\delta},\quad (z\in\mathcal{U})$$

where $1/2\leq\delta<1$.

Thus, this Corollary reduces to the result shown in the ([2], Corollary 2.4).

Theorem 2.7: Let $0\leq\rho<1$ and $0\leq\mu<1$. If $f(z)\in M^*$ satisfies

$$\left|(1-\mu)+\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)''}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}+\mu\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}-\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}-\gamma\left(\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}\right)\right|$$

$$<\frac{(1-\rho)(\gamma(2-\rho)+1)}{2-\rho}.\qquad (12)$$

Then $f(z)\in\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\rho)$

Proof. Define $w(z)$ by

$$\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}=-1+(\rho-1)w(z),\qquad (13)$$

then $w(z)$ is analytic function and $w(0)=0$.Differentiating logarithmically of (13) with respect to z, we get

$$(1-\mu)+\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)''}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}+\mu\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)}-\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}=\frac{(1-\rho)zw'(z)}{1+(1-\rho)w(z)}.\qquad (14)$$

Using (13) in (14), we get

$$(1-\mu)+\frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))''}{(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'}+\mu\frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}-\frac{z(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}-\gamma\left(\frac{z^{1-\mu}(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))'(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z))^{\mu}}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}\right)$$

$$=\gamma(1-\rho)w(z)+\frac{(1-\rho)zw'(z)}{1+(1-\rho)w(z)}.\qquad (15)$$

Let $z_0\in\mathcal{U}$ such that

$$\max_{|z|<|z_0|}|w(z)|=|w(z_0)|,$$

and application Lemma 1.3, we get

$$z_0w(z_0)=kw(z_0),\qquad (k\geq 1)$$

setting $w(z)=e^{i\theta}$ $(0\leq\theta<2\pi)$ and putting $z=z_0$ in (15), we have

$$\left|(1-\mu)+\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)''}{\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}+\mu\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}-\frac{z\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}-\gamma\left(\frac{z^{1-\mu}\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_\lambda^{m,\beta}(\alpha,\xi)f(z)}\right)\right|=\left|\gamma(1-\rho)e^{i\theta}+\frac{(1-\rho)ke^{i\theta}}{1+(1-\rho)e^{i\theta}}\right|\quad (16)$$

$$\geq Re\left(\gamma(1-\rho)+\frac{(1-\rho)k}{1+(1-\rho)e^{i\theta}}\right)$$

$$>\gamma(1-\rho)+\frac{(1-\rho)}{2-\rho}$$

$$=\frac{(1-\rho)(\gamma(2-\rho)+1)}{2-\rho},$$

which contradicts our assumpion (12). Therefor, we have $|w(z)|<1$ in $\mathcal{U}$. Finally, we have

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$$\left|\frac{z^{1-\mu}\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}+1\right|$$

$$=|(1-\rho)w(z)|=(1-\rho)|w(z)|<1-\rho. \qquad (17)$$

Note that the condition (17) is equivalent to

$$-Re\left\{\frac{z^{1-\mu}\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)'\left(\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)\right)^{\mu}}{\Omega_{\lambda}^{m,\beta}(\alpha,\xi)f(z)}\right\}>\rho,$$

that is $f(z)\in\mathcal{R}_{\lambda,\alpha}^{m,\beta}(\xi,\mu,\rho)$.

For $\mu=0, m=0, \rho=0$ and $\gamma=0$ in Theorem 2.7, we have.

Corollary 2.8: If $f(z)\in M^*$ satisfies the following inequality

$$\left|\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+1\right|<\frac{1}{2},$$

then $f(z)\in M_{s^*}^*$.

For $\mu=1, m=0$ and $\gamma=1$ in Theorem 2.7, we obtain.

Thus, this Corollary reduces to the result shown in

the [[7], Corollary 7] .

Corollary 2.9: If $f(z)\in M^*$ satisfies the following inequality

$$\left|\frac{zf''(z)}{f'(z)}-f'(z)\right|<\frac{(1-\rho)(3-\rho)}{2-\rho},$$

then $f(z)\in\mathcal{R}_{\lambda,\alpha}^{0,\beta}(\xi,1,\rho)$.

Further, putting $\rho=0$ in Corollary 2.9, we get

Corollary 2.10: If $f(z)\in M^*$ satisfies the following inequality

$$\left|\frac{zf''(z)}{f'(z)}-f'(z)\right|<\frac{3}{2},$$

then $f(z)\in\mathcal{R}_{\lambda,\alpha}^{0,\beta}(\xi,1,0)$.

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